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Cohomology of quasiperiodic patterns and matching rules

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Abstract

Quasiperiodic patterns described by polyhedral 'atomic surfaces' are considered. It is shown that under certain rationality conditions (which coincide with the necessary conditions for the existence of matching rules), the cohomology ring of the continuous hull of such patterns is isomorphic to that of the complement of a torus T^N to an arrangement A of thickened affine tori of codimension 2. Explicit computation of Betti numbers for several two-dimensional tilings and for the icosahedral Ammann–Kramer tiling confirms in most cases the results obtained previously by different methods. The cohomology groups of $T^N \backslash A$ have a natural structure of a right module over the group ring of the space symmetry group of the pattern and can be decomposed in a direct sum of its irreducible representations. An example of such decomposition is shown for the Ammann–Kramer tiling.

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1. Introduction

One of the distinct features of crystalline structures is that they are characterized by discrete parameters, in addition to continuous ones. Examples of such discrete parameters are lattice symmetry classes, numbers of atoms in the unit cell, occupancies of Wyckoff positions, etc. Taking into account the role of discrete parameters in our understanding of the structure, it is appealing to find similar parameters for quasicrystals. Certain of them could be obtained as a generalization of the discrete parameters specific for crystals in the framework of the 'cut-and-project' model. This is the case, e.g., for the symmetry class of the underlying high-dimensional lattice [1] or for the number of atoms in the unit cell (which is replaced by the homology class of the atomic surface [2]). The efforts to develop a more systematic approach to the problem have lead to a promising concept of mutual local derivability (MLD) [3, 4].

An alternative approach to classification of quasicrystals is based on the notion of the *hull* of a quasiperiodic structure. The concept of hull originated from the works by Bellissard [5]

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on C^*-algebras of observables in solid-state physics. Bellissard conjectured that this algebra includes a crossed product of the algebra of functions on a topological space (called 'the hull') with the group of translations acting on it. In many cases, including the one-particle Schrödinger equation in a quasiperiodic potential, the hull can be described explicitly. The quasiperiodic patterns of the same MLD class have homeomorphic hulls, which allows one to characterize quasicrystals by algebraic topological invariants of their hulls. In this paper, we show that some of these invariants, namely the cohomology ring of the hull may also occur in the study of the matching rules of quasicrystals.

Before proceeding any further, let us describe briefly the geometric constructions used in the paper. Following the so-called 'cut-and-project' method, a quasiperiodic point set is obtained as an intersection of d-dimensional affine subspace $E_\parallel \subset \mathbb{R}^N$ with a periodic arrangement of $(N-d)$-dimensional manifolds (with boundary) in $\mathbb{R}^N$. The space $E_\parallel$ is usually referred to as a 'physical space', or 'cut', and the $(N-d)$-dimensional manifolds are called 'atomic surfaces'. One can define affine coordinates on $\mathbb{R}^N$ in such a way that the periodic translations of the arrangement of atomic surfaces correspond to the vectors with integer coefficients. The space $\mathbb{R}^N$ can be factored by integer translations, yielding the N-dimensional torus T^N. We also assume that $E_\parallel$ is not contained in any proper rational subspace of $\mathbb{R}^N$, hence its image under the natural projection $\pi : \mathbb{R}^N \to T^N$ fills densely the torus T^N.

In this paper, we consider polyhedral atomic surfaces only. In order to simplify the proofs we also make several other non-essential assumptions. In particular, we require that all connected parts of the atomic surface be flat and parallel to an $(N-d)$-dimensional affine subspace $E_\perp \subset \mathbb{R}^N$. The $\mathbb{R}^N$ is furnished with a Euclidean metric, such that $E_\parallel$ and $E_\perp$ are perpendicular. When this does not lead to confusion, we will implicitly switch between $\mathbb{R}^N$ and T^N. In particular, we will use symbols $E_\parallel$ and $E_\perp$ to designate subspaces in the local coordinate system on T^N. The term 'atomic surface' will also signify the submanifold $S \subset T^N$ obtained by the natural projection of atomic surfaces from $\mathbb{R}^N$. Likewise, we will speak about translations and convolutions in T^N implying the operations in the universal cover of T^N. The same applies to the definition of 'piecewise-linear' (PL) subspaces of T^N.

2. Matching rules, obstacles and the rationality condition

In this section, we describe the construction of the arrangement of thickened affine tori. This arrangement arises the most naturally in the study of the matching rules of quasiperiodic patterns. As we shall see, this construction is only possible if the boundary of the atomic surface satisfies certain rationality condition, which is also a prerequisite to the existence of the strong matching rules.

The term 'matching rules' is usually taken to mean the set of local constraints on a pattern (a tiling or a discrete set of points) guaranteeing its global quasiperiodicity. One can distinguish two approaches to the construction of matching rules. One approach, which was historically the first, is based on the scaling symmetry of the quasiperiodic pattern [6, 7]. The other one is built upon a more physical idea of propagation of the quasiperiodic order and leads to the topological formulation of the matching rules [8, 9, 11]. Let us briefly recall the derivation of the latter approach.

From the very beginning of the study of quasicrystals it has become obvious that their stability is closely related with the possibility of propagation of information about the local phason coordinate. In particular, the stability requires that the places at which the structure undergoes reconstruction under a uniform phason shift be arranged in a special way. Namely, when the magnitude of the phason shift tends to zero, the minimal distance between the

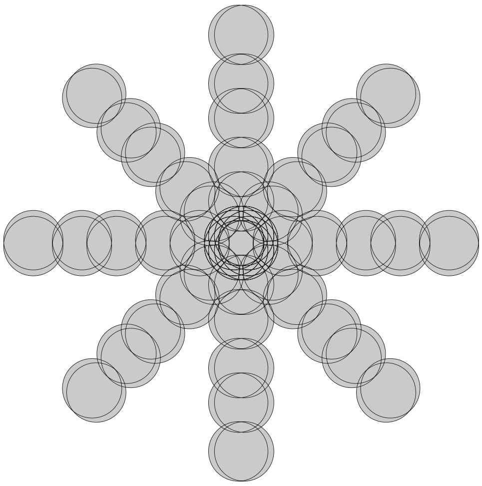

Figure 1. The globally connected net formed by R-discs centred at the points where a singular cut crosses the boundary of the atomic surface of an undecorated Ammann octagonal tiling. This cut passes through the vertices of the atomic surface.

places where the structure is rearranged should not grow indefinitely, because otherwise no physical mechanism could guarantee the simultaneousness of the rearrangements [12]. More precisely, there should exist such positive number R, that the union of discs of radius R, centred at the places where the rearrangements occur, form a globally connected net for any finite uniform phason shift (see figure 1). In the general case, the geometry of this net could be quite complicated. However, we shall restrict our consideration to an important special class of structures described by flat atomic surfaces with polygonal boundary. This class includes in particular the so-called model sets [13]. in this case, the rearrangements of atoms under a uniform phason shift occur only when the cut crosses the boundary ∂S of the atomic surface; this boundary thus plays a crucial role in the propagation of the quasiperiodic order. In particular, it can be shown that the matching rules impose certain constraints on the orientation of the faces F_i of the boundary [9, 12]. Namely, the orientation of each face F_i is such that a *singular cut*, crossing it at one point, will cross it at an infinite set of points, which is dense in F_i. When considered in the space of the cut, these points form an R-dense set [10] in a hyperplane in $E_\parallel$, as can be seen in figure 1. It is convenient to associate with each face F_i of the atomic surface a pair of unit vectors $\mathbf{k}_i \in E_\perp$ and $\mathbf{n}_i \in E_\parallel$, which are respectively normal to F_i and to the hyperplane in $E_\parallel$ containing the points of intersection of the singular cut with F_i. As F_i is crossed by the singular cut at a dense set of points, this is also true for the extension of F_i (i.e., the intersection of all affine subspaces of R^N containing F_i). The closure of this extension in the topology of T^N is clearly an affine torus, which is orthogonal to both $\mathbf{k}_i \in E_\perp$ and $\mathbf{n}_i \in E_\parallel$ and intersects $E_\parallel$ and $E_\perp$ at subspaces of codimension 1. We are now at the point to formulate the rationality condition on the boundary of the atomic surface:

Rationality condition. *The atomic surface is called to satisfy the rationality condition if for any of its faces F_i there exist two vectors $\mathbf{n}_i \in E_\parallel$ and $\mathbf{k}_i \in E_\perp$ such that the closure of the extension of the face F_i in topology of T^N is an affine torus of dimension $N-2$, perpendicular to $\mathbf{n}_i$ and $\mathbf{k}_i$.*

Hereinafter, we assume that the atomic surface satisfies this condition. It is worth mentioning here that this condition is only necessary for the existence of strong matching

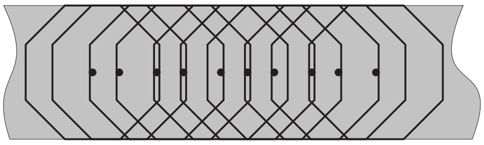

Figure 2. Thickened hyperplane.

rules. In fact, there are quasiperiodic patterns which do not admit strong matching rules (e.g., the non-decorated octagonal and dodecagonal tilings), but for which the rationality condition is satisfied and hence the cohomology of the continuous hull still can be computed by the method described in this paper.

It is important to note that the rearrangements of the quasiperiodic pattern under the action of the uniform phason shift occur simultaneously on the entire net of the figure 1. Since such rearrangements do not break the perfect quasiperiodic order, the matching faults may occur only at the places where the synchronization of rearrangements is broken at distances smaller than some finite R. The idea of the topological description of the matching rules stems from an observation that such defects can be produced if one allows the cut to undulate. in this case, the matching faults would correspond to intersections of the undulating cut with the set $Y_R = \partial S + B_R^{\parallel}$, where $B_R^{\parallel}$ stands for an R-ball in the parallel direction. The set Y_R is naturally referred to as 'obstacles' of 'forbidden set' [8, 9]. The obstacles Y_R are said to define strong matching rules if any map of the physical space into $T^N \backslash Y_R$, satisfying some mild 'sanity conditions' (e.g., to be everywhere transversal to the direction of $E_\perp$), is homotopy equivalent to a perfect cut.

Let us take a closer look at the obstacles Y_R in the case when the atomic surface satisfies the rationality condition. The set Y_R can be represented as

$$Y_R = \bigcup_{i \in I} t_{R,i}, \tag{1}$$

where the set I enumerates the faces of the atomic surface and $T_{R,i}$ is defined as

$$t_{R,i} = F_i + B_R^{\parallel}. \tag{2}$$

From this point on, it is convenient to replace the Euclidean norm used to define the R-balls in the parallel space by the following equivalent one. For a vector $\mathbf{v} \in E_\parallel$ this norm is given by

$$\|\mathbf{v}\| = \max_{i \in I} \left(|\mathbf{n}_i \cdot \mathbf{v}|\right), \tag{3}$$

where the index i enumerates the faces of the atomic surface. Note that the expression (3) may not define a norm if the vectors $\{\mathbf{n}_i\}$ span a proper subspace of $E_\parallel$. If this is the case, we can turn (3) into a norm by appending to $\{\mathbf{n}_i\}$ the vectors of a basis of the orthogonal complement to this subspace. The advantage of the norm (3) over the ordinary Euclidean one is that the set Y_r defined with the former has an especially simple geometry. To see this, consider the intersection of a singular cut with the set $t_{r,i}$ (2). This intersection is a union of r-balls centres with belonging to an R-dense set on a hyperplane perpendicular to $\mathbf{n}_i$. Note also that an r-ball defined with the norm (3) is a convex polyhedron and two of its faces are perpendicular to $\mathbf{n}_i$. As is clear from figure 2, the union of such r-balls for r big enough is a 'thickened' hyperplane (a set of points $\mathbf{x} \in E_\parallel$ satisfying $a - r \leqslant \mathbf{x} \cdot \mathbf{n}_i \leqslant a + r$ for some a). This can only be possible if the set $t_{r,i}$ takes the form of a 'thickened torus':

$$t_{r,i} = T_i + I_i, \tag{4}$$

where T_i is an affine subtorus of T^N of codimension 2 orthogonal to both $\mathbf{n}_i$ and $\mathbf{k}_i$, I_i is a segment of length $2r$ parallel to $\mathbf{n}_i$ and the sign '+' stands for the convolution (see remark in the introduction). In what follows we will frequently use the notion of thickened affine torus, and it is convenient to give it a broader definition, which will include (4) as a special case:

Definition. *A thickened affine torus t is a convolution of an affine torus T with a compact convex subset B of $E_\parallel$:*

$$t = T + B. \tag{5}$$

Thus, we have shown that for r big enough, the obstacle Y_r is a finite union of thickened affine tori (5). Note also that Y_r can be equipped with a Whitney stratification [14] in such a way that any thickened torus containing a point of a stratum contains the entire stratum.

3. Equivalence of cohomology rings of the continuous hull and $T^N\backslash A$

Following [15], we define the continuous hull MP of the quasiperiodic pattern as a completion of set of punctured patterns in the metric of 'approximate match' D (roughly speaking, two patterns are separated by the distance $<\epsilon$ in the metric D if within the ball of radius $1/\epsilon$ the Hausdorff distance between them is smaller than ϵ; for exact definition see [15]). In this section, we establish the equivalence of the cohomology ring of MP and that of a complement of T^N to an arrangement of thickened affine subtori A. We start the proof by constructing a sequence of topological spaces X_r parametrized by a real r, and show that MP is homeomorphic to the inverse limit of this sequence. Then we show that in the case of quasiperiodic patterns admitting matching rules, in the homotopy category, the limit is attained for a finite r_0. Finally, we demonstrate that the space X_{r_0} is homotopy equivalent to $T^N\backslash A$. Note that the representation of the continuous hull of a quasiperiodic pattern as an inverse limit of topological spaces has already been used in the literature [16, 17]. Unlike the above references, the present approach deals directly with the cut-and-project representation of the quasiperiodic pattern, which allows for a more intuitive description of the limit space. It should also be mentioned that the role of the matching rules in convergence of the sequence of cohomology groups of approximating spaces has been conjectured in [18].

The set $T^N\backslash Y_r$ represents the origins of the cuts producing non-singular patterns at least within the r-disc centred at the origin. In order to include the singular patterns, one has to add some more points to this space, which could be done by considering a metric closure of $T^N\backslash Y_r$. Let us start with the metric on T^N induced by the Euclidean metric of R^N after factoring it over $\mathbb{Z}^N$ in the standard position. It induces an inner metric on $T^N\backslash Y_r$ [19] (in this metric the distance between two points equals the infimum of the lengths of the paths in T^N connecting them and avoiding Y_r). Denote the completion of $T^N\backslash Y_{r_n}$ with respect to this metric by X_{r_n}. Consider now an unbounded monotonously increasing sequence r_n and the inclusion maps $\iota'_n : T^N\backslash Y_{r_{n+1}} \to T^N\backslash Y_{r_n}$. Because ι'_n do not increase the distance between points, these maps can be extended to X_{r_n}:

$$\iota_n : X_{r_{n+1}} \to X_{r_n}. \tag{6}$$

One can define the inverse limit of the maps (6)

$$X = \lim_{\leftarrow} X_{r_n}$$

together with the corresponding projections $\pi_n : X \to X_{r_n}$.

Corollary 1. *The space X is homeomorphic to the continuous hull MP defined in [15].*

Proof. Recall that MP is defined as completion of the space of non-singular patterns with respect to the metric of 'approximate match' D of [15] (two patterns have a distance lesser than ϵ if the Hausdorff distance between their patches of size $1/\epsilon$ does not exceed ϵ). First of all, remark that there exist continuous maps $\xi_i : MP \to X_i$, satisfying $\xi_n = \iota_n \xi_{n+1}$:

$$\begin{tikzcd} \cdots \arrow[r, "\iota_{n+1}"] & X_{r_{n+1}} \arrow[r, "\iota_n"] & X_{r_n} \arrow[r, "\iota_{n-1}"] & \cdots \arrow[r, "\iota_0"] & X_{r_0} \\ & MP \arrow[u, "\xi_{n+1}"] \arrow[ur, "\xi_n"] \arrow[urrr, "\xi_0"] & & & \end{tikzcd} \tag{7}$$

To define the maps ξ_i, consider a point $a \in$ MP. This point is a limit of a sequence of patterns obtained by nonsingular cuts, which is Cauchy in the metric of 'approximate match'. The origins of these cuts form a sequence of points $x_n \in$ NS, where NS $= \bigcap_i \left(T^N \backslash Y_{r_i}\right)$. In the metric of T^N the sequence x_n converges to a point $w \in T^N$ (which may belong to a singular cut!). Consider the cuts with the origins belonging to the $w + B_\epsilon^\perp$, where $B_\epsilon^\perp$ is an open PL ϵ-ball in $E_\perp$. Those of them, which cross ∂S at the distance less or equal to r_k from the origin, divide $B_\epsilon^\perp$ in a finite number of open polyhedral pieces c_j. There exist n_ϵ and j_0 such that for $n > n_\epsilon$ all points x_n belong to $w + c_{j_0} \times B_\epsilon^\parallel$, where $B_\epsilon^\parallel$ is an ϵ-ball in $E_\parallel$. Consider any two points x_{n_1} and x_{n_2} of the sequence for which $n_1, n_2 > n_\epsilon$. Since $c_{j_0} \times B_\epsilon^\parallel$ does not intersect Y_{r_k}, the distance between them in the induced inner metric of $T^N \backslash Y_{r_k}$ is bounded by const $\cdot\, \epsilon$. Therefore, the sequence x_n is Cauchy in the latter metric in $T^N \backslash Y_{r_k}$ and converges to a point in X_{r_k}, which we set as $\xi_k(a)$. The continuity of ξ_k and commutativity of (7) are obvious.

Consider now the continuous map ζ : MP $\to X$, satisfying $\pi_i \zeta = \xi_i$, which exists by virtue of the universal property of inverse limits. Since ξ_n separates any two points $a, b \in$ MP for which $D(a, b) > 1/r_n$, the map ζ is injective. To establish the surjectivity of ζ, consider a point $x \in X$. For each k, its image $\pi_k(x)$ can be approximated by a sequence of points $x_{k,i} \in$ NS $\subset T^N \backslash Y_{r_k}$:

$$\lim_{i \to \infty} x_{k,i} = \pi_k(x).$$

The convergence here is defined in the metric of $T^N \backslash Y_{r_k}$ and without loss of generality can be assumed to be uniform in k. The inclusions NS $\subset T^N \backslash Y_{r_n} \subset X_{r_n}$ allows one to consider $x_{k,i}$ as a point in X_{r_n} for any n. Then the diagonal sequence $y_i = x_{i,i} \in$ NS converges in each X_{r_k} to $\pi_k(x)$ (this follows from the fact that the maps ι_n of (6) do not increase distance between points). The patterns obtained by cuts with origins at the points y_i form a Cauchy sequence in the metric of 'approximate match'. The limit of this sequence is a point in MP which we set as $\zeta^{-1}(x)$. Therefore, the map ζ is a continuous bijection of a compact Hausdorff space MP [15], and hence a homeomorphism. □

The consideration in section 2 suggests that the homotopy type of $T^N \backslash Y_r$ stabilizes with increasing r, and one would expect the same for X_r. This is indeed the case, more precisely, for the polygonal atomic surfaces the following result holds (the proof is given in appendix):

Corollary 2. *There exists an arrangement A of thickened affine subtori of T^N and a finite positive r, such that for any $r_{n+1} > r_n \geqslant r$ there is an inclusion $A \subset Y_{r_n}$ and the following maps are homotopy equivalences:*

(i) The natural inclusion $\mu_n : T^N \backslash Y_{r_n} \to X_{r_n}$.
(ii) The inclusion of complements $\nu_n : T^N \backslash Y_{r_n} \to T^N \backslash A$.
(iii) The map $\iota_n : X_{r_{n+1}} \to X_{r_n}$ from (6).

An immediate consequence of the above Corollary follows is that the homomorphisms of cohomology rings induced by (6)

$$\iota_n^* : H^*(X_{r_n}) \to H^*(X_{r_{n+1}})$$

are isomorphisms for $r_n \geqslant r$. Thus the cohomology ring of the space X equals that of $T^N \backslash A$:

$$H^*(X) = \lim_{\rightarrow}(H^*(X_{r_n})) = H^*(X_r) = H^*(T^N \backslash A). \tag{8}$$

Combining (8) with corollary 1 we conclude that the cohomology ring of the continuous hull MP of a quasiperiodic pattern admitting matching rules is isomorphic to that of a complement of T^N to a finite arrangement of thickened affine tori of codimension 2. This implies in particular that the cohomologies of MP are finitely generated and can be explicitly calculated as discussed below.

4. Cohomology of $T^N \backslash A$

Our goal is to find the cohomology groups of the complement of the N-dimensional torus to an arrangement of thickened affine tori A. Let us start with the exact cohomological sequence of pair $(T^N, T^N \backslash A)$

$$\begin{tikzcd} H^*(T^N, T^N\backslash A) \arrow[r, "\alpha^*"] & H^*(T^N) \arrow[d, "\beta^*"] \\ & H^*(T^N\backslash A) \arrow[ul, "d^*"] \end{tikzcd} \tag{9}$$

As a Whitney stratified subspace of a torus, A can be surrounded by an open mapping cylinder neighbourhood $\tilde{A}$ [20]. The mapping cylinder determines a deformation retraction of $\tilde{A}$ onto A as well as that of $T^N \backslash A$ onto $T^N \backslash \tilde{A}$. As T^N is a compact manifold and $T^N \backslash \tilde{A}$ is its closed subspace, one has from Poincaré–Alexander–Lefschetz duality [21]

$$H^i(T^N, T^N \backslash \tilde{A}) = H_{N-i}(\tilde{A}) \tag{10}$$

giving due to the deformation retraction property

$$H^i(T^N, T^N \backslash A) = H_{N-i}(A). \tag{11}$$

The long exact sequence (9) together with the duality relation (11) links the cohomologies of $T^N \backslash A$ with the homologies of A. This is not yet sufficient to relate $H_{N-i-1}(A)$ with $H^i(T^N \backslash A)$ in each dimension (this would be the case if the homologies of the surrounding space vanished in several adjacent dimensions, as is the case for a sphere, yielding Alexander duality). However, if the rank of the homomorphism β^* is known, it is still possible to separate the dimensions in the sequence (9). Indeed, (9) could be split in five-term exact sequences:

$$0 \longrightarrow \mathrm{Im}(\beta^{n-1}) \longrightarrow H^{n-1}(T^N \backslash A) \xrightarrow{d^n} H_{N-n}(A) \xrightarrow{\alpha^n} H^n(T^N) \longrightarrow \mathrm{Im}(\beta^n) \longrightarrow 0\ , \tag{12}$$

yielding the following equation on Betti numbers:

$$b_{n-1}(T^N \backslash A) = b_{N-n}(A) + c_{n-1} + c_n - \binom{N}{n}, \tag{13}$$

where c_n stands for the rank of the map

$$\beta^n : H^n(T^N) \rightarrow H^n(T^N \backslash A). \tag{14}$$

Thus, the ranks of cohomology groups of $T^N \backslash A$ are determined by that of the homology groups of A and the ranks of the maps (14). To obtain the latter remark that by exactness of (12), the kernel of β^n is isomorphic to the image of α^n. On the other hand, α^n is by Poincaré duality equal to the map $H_{N-n}(A) \rightarrow H_{N-n}(T^N)$ induced by inclusion $A \subset T^N$.

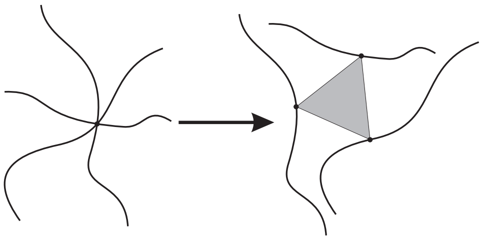

Figure 3. The simplicial resolution of the triple intersection of one-dimensional manifolds. The intersection point is replaced by a contractible space (here a two-dimensional simplex) and the multiplicity of intersection is lowered from 3 to 2.

5. Homology of an arrangement of affine tori

The space A defined in corollary 2 is in general case an arrangement of *thickened* affine tori. However, as mentioned in the appendix, in many cases this space can be collapsed to an arrangement of ordinary affine tori, which simplifies the computation significantly. In this section, we assume that A is already collapsed to such an arrangement.

The homology groups of an arrangement of affine tori could be conveniently computed using the method of simplicial resolutions (see, e.g., [22], although we follow here a slightly modified version of the method) and by consecutive application of Mayer–Vietoris spectral sequence. With this technique, instead of the arrangement A, one considers its *resolution space* A^{Δ}, which has the same homotopy class as A (an example of simplicial resolution is shown in figure 3). The explicit construction of A^{Δ} is as follows. Let us associate with the arrangement A a combinatorial object $L(A)$ called an *intersection poset*. The elements of the intersection poset $x \in L(A)$ correspond to connected components of nonempty intersections of the tori constituent the arrangement A, and the partial order is given by reverse inclusion. Note that each nonempty intersection of affine tori is itself a disjoint union of affine tori (we treat a point as a special case of zero-dimensional torus). Consider an abstract simplex Δ with vertices enumerated by maximal chains of $L(A)$. For each $\in L(A)$, the maximal chains containing x define a face of Δ, which we denote by Δ_x. Let also $t_x \subset T^N$ stand for the affine torus corresponding to x. Then the space of the simplicial resolution of A is defined as

$$A^{\Delta} = \bigcup_{x \in L(A)} t_x^{\Delta}, \tag{15}$$

where

$$t_x^{\Delta} = t_x \times \Delta_x \tag{16}$$

and the corresponding projection $h : A^{\Delta} \to A$ is induced by the projection of $T^N \times \Delta$ onto the first component.

Corollary 3. *The projection $h : A^{\Delta} \to A$ is a homotopy equivalence.*

Proof. First of all, let us show that for any point $a \in A$, the space $h^{-1}(a)$ is contractible. By construction, $h^{-1}(a)$ is a simplicial set:

$$h^{-1}(a) = a \times \bigcup_{y \in L_a} \Delta_y, \tag{17}$$

where $L_a = \{y \in L | a \in t_y\}$. Note that there exists a maximal element $x \in L_a$ defined by the condition $t_x = \bigcap_{y \in L_a} t_y$. Obviously, for any subset $\{y_i\} \subset L_a$ satisfying $\bigcap_i \Delta_{y_i} \neq \varnothing$ the elements y_i form a chain, which can always be extended by including x. In other words, any non-empty intersection of simplices Δ_y in (17) contains at least one vertex of Δ_x. Consider a vertex $v \in \bigcup_{y \in L_a} \Delta_y$, which does not belong to Δ_x. The intersection of all simplices Δ_y containing v is nonempty and thus contain at least one vertex $v' \in \Delta_x$ and hence the entire edge $[vv']$. Collapsing $[vv']$ towards v' defines a deformation retraction of the entire simplicial set (17) onto its subset obtained by removing the vertex v. This operation can be repeated to eliminate other vertices not belonging to Δ_x, which proves the contractibility of (17).

Recall now that A is a Whitney stratified space. By construction, the set L_a does not depend on the position of the point a in the stratum. In other words, over each stratum, the resolution space A^Δ has a structure of a trivial bundle with contractible layer. This observation enables us to follow the proof of lemma 1 from [22], section 3.3.3. Namely, consider a triangulation of A, which exists due to [23]. The interior of each simplex σ of triangulation is contained within a stratum. Hence, the space $h^{-1}(\sigma)$ also has a structure of trivial bundle with a contractible layer. Then the projection h can be decomposed as

$$h = h_n \circ \cdots \circ h_1 \circ h_0, \tag{18}$$

where h_k contracts the layers over the interior points of k-dimensional simplices of the triangulation (h_k are continuous because the layers over the boundary of the simplex are already contracted). The maps h_k from (18) are homotopy equivalences, which proves that h is also a homotopy equivalence. □

At the first glance, the simplicial resolution only replaces an arrangement of tori by the union (15) of a bigger number of more complex objects (16). However, these objects intersect each other in a more simple way. In particular, $t_x^\Delta \cap t_y^\Delta$ is nonempty iff x and y are comparable. In a similar manner, several spaces (16) have nonempty intersection iff the corresponding elements of $L(A)$ form a chain. In this case, the intersection has the form

$$\bigcap_i t_{y_i}^\Delta = t_{\max(y_i)} \times \delta, \tag{19}$$

where δ is a face of Δ. Because the comparable elements in $L(A)$ correspond to tori of different dimensions, the maximal number of intersecting spaces $t_{y_i}^\Delta$ in (19) cannot exceed $N+1$. As we shall see below, this limits the number of non-zero columns in the corresponding Mayer–Vietoris double complex to $N + 1$ (actually this number is even smaller—it equals 2 for two-dimensional patterns and 3 for the icosahedral Ammann–Kramer tiling).

The final step in the computation of the homology groups of the arrangement of affine tori consists of application of Mayer–Vietoris spectral sequence to the resolution space obtained above. We use the homology version of the bi-complex described, e.g., in chapter 2 of [24]. Namely, let us consider a finite CW-space X, which is a union of finite number of CW-spaces:

$$X = \bigcup_\alpha U_\alpha.$$

The groups of chains of the Mayer–Vietoris bi-complex are defined as

$$C_{p,q} = \bigoplus_{\alpha_1,\cdots,\alpha_p} C_q\left(U_{\alpha_0,\ldots,\alpha_p}\right),$$

where $U_{\alpha_0,\ldots,\alpha_p}$ stands for a $(p+1)$-wise intersection

$$U_{\alpha_0,\ldots,\alpha_p} = \bigcap_{i=0}^{p} U_{\alpha_i}$$

and C_q are ordinary q-chains (e.g., singular ones). The differential $\partial : C_{p,q} \to C_{p,q-1}$ is the ordinary boundary operator multiplied by $(-1)^p$, and the differential $\delta : C_{p,q} \to C_{p-1,q}$ is defined as

$$\delta c_q\big(U_{\alpha_0,\ldots,\alpha_p}\big) = \sum_{n=0}^{p}(-1)^n c_q\big(U_{\alpha_0,\ldots,\hat{\alpha}_n,\ldots,\alpha_p}\big),$$

where the hat denotes omission (in this formula the same chain c_q is considered as belonging to both $(p+1)$-wise and p-wise intersections). There are two spectral sequences associated with this bi-complex [24]; we choose the one starting with the homology groups with respect to the differential ∂:

$$E^1_{p,q} = H_\partial(C_{p,q}).$$

As there is a finite number of non-zero columns in the original bi-complex, this sequence converges at a finite step and yields the graded complex associated with the homology groups of X.

6. Two-dimensional patterns

In the case of two-dimensional quasiperiodic patterns satisfying the rationality conditions the space A is an arrangement of two-dimensional affine subtori of a four-dimensional torus. As we shall see, in all cases of interest, these tori intersect each other transversally, that is at a discrete set of points. Let m denote the number of tori in A. We also denote by n_k the number of points at which k affine tori intersect simultaneously. The simplicial resolution of A yields m spaces which are homotopy equivalent to two-dimensional tori and $\sum_k n_k$ simplices. All intersections between these spaces are pairwise, giving $\sum_k kn_k$ intersection points. The only non-zero groups in the term E^1 of the homology spectral sequence of the corresponding Mayer–Vietoris double complex are the followings:

$$E^1_{0,2} = \mathbb{Z}^m \qquad E^1_{0,1} = \mathbb{Z}^{2m} \qquad E^1_{0,0} = \mathbb{Z}^{m+\sum_k n_k} \qquad E^1_{1,0} = \mathbb{Z}^{\sum_k kn_k}. \tag{20}$$

Since the above spectral sequence has only two non-zero columns, it collapses at the E^2-term. The only nontrivial differential between the groups (20) is $\delta : E^1_{1,0} \to E^1_{0,0}$. The rank of this differential equals $m + \sum_k n_k - p$, where p stands for the number of connected components of A. This yields the following Betti numbers of A:

$$b_2(A) = m \qquad b_1(A) = m + p + \sum_k (k-1)n_k \qquad b_0(A) = p. \tag{21}$$

To obtain the Betti numbers of $T^N\backslash A$, one also needs to know the ranks c_n of the maps β^n (14). Since A does not contain cells of dimension higher than 2, the maps β^0 and β^1 are injective, giving $c_0 = 1$ and $c_1 = 4$. On the other hand, in all cases considered below, any 0-cycle and 1-cycle on T^4 can be represented by a cycle on A. Therefore α^3 and α^4 from (12) are surjective, yielding $c_3 = 0$ and $c_4 = 0$. To obtain the rank of the remaining map $\beta^2 : H^2(T^4) \to H^2(T^4\backslash A)$ observe that since $E^1_{11} = 0$ and $E^1_{20} = 0$, the group $H_2(A)$ is the direct sum of the groups H_2 of two-dimensional tori constituent A. This allows for explicit computation of the image of α^2 (12). In all cases considered below except of undecorated Ammann–Beenker tiling and undecorated dodecagonal tiling the rank of α^2 equals 4, which corresponds to $c_2 = 2$. This result is likely to be valid for any two-dimensional quasiperiodic pattern admitting strong matching rules, because of the following argument using de Rham cohomologies. The volume forms $\omega_\parallel$ and $\omega_\perp$ in $E_\parallel$ and $E_\perp$ are closed 2-forms on T^4 spanning a two–dimensional space in $H^2_{\mathrm{DR}}(T^4)$. On the other hand, one can embed $\mathbb{R}^2$ in T^4 in directions

Table 1. Betti numbers of $T^N\backslash A$ for various two-dimensional quasiperiodic patterns. In addition to Betti numbers b_1 and b_2 the following parameters of the arrangement A are given: the number of tori m, the number of connected components p, the rank c_2 and the numbers of k-wise intersection points n_k. These parameters enter in formulae (21) and (13).

Tiling	b_1	b_2	m	p	c_2	Numbers of intersections
Ammann–Beenker	5	9	4	1	3	$n_2 = 2, n_4 = 1$
Ammann–Beenker decorated	8	23	8	1	2	$n_2 = 6, n_4 = 1, n_8 = 1$
Penrose ($\gamma \in \mathbb{Z}[\tau]$)	5	8	5	1	2	$n_5 = 1$
Penrose (γ generic)	10	34	10	1	2	$n_2 = 10, n_4 = 5$
Dodecagonal	7	28	6	1	3	$n_2 = 9, n_3 = 4, n_6 = 1$
Dodecagonal decorated	12	59	12	1	2	$n_2 = 12, n_3 = 8, n_4 = 3, n_{12} = 1$

of either $E_\parallel$ or $E_\perp$ without intersecting A. This suggests that $\beta^2(\omega_\parallel) \neq 0$ and $\beta^2(\omega_\perp) \neq 0$, that is the rank of β_2 is at least equal to 2. On the other hand, the rank of β^2 cannot be bigger than 2, because this would allow for continuous variation of the 'slope' of $E_\parallel$ in $T^N\backslash A$, which is forbidden by the matching rules. Indeed, the n-dimensional volume forms in R^N are parametrized by the points of the Grassmann manifold $g_{N,n}$. Since $\dim(g_{4,2}) = 4$, the manifold of volume forms has codimension 2 in $H^2_{\mathrm{DR}}(T^4)$. If the dimension of $\mathrm{Im}(\beta^2)$ equals 3, this space would intersect the above manifold in the general case along one-dimensional curves, which would make possible a continuous variation of the slope of $E_\parallel$. One can cite as an example the undecorated versions of octagonal Ammann–Beenker and dodecagonal tilings, for which the rank of β_2 equals 3, and which do not admit matching rules.

Let us illustrate the technique described above by calculating the Betti numbers for the Ammann–Beenker octagonal tiling. The 'atomic surface' of this tiling in its undecorated version has the shape of a perfect octagon. Eight edges of the octagon give rise to eight thickened affine tori (4). However, the tori corresponding to the opposite edges knit together as r increases. This results in four thickened tori, which have a nonempty intersection and thus can be collapsed to four affine tori t_i. They could be specified by the following vectors spanning the corresponding hyperplanes in the universal covering space of T^4:

$$t_1 : (e_1, e_2 - e_4) \qquad t_2 : (e_2, e_1 - e_3) \qquad t_3 : (e_3, e_2 + e_4) \qquad t_4 : (e_4, e_3 - e_1),$$

and by the condition that they all pass through the origin. Here e_i stand for the basis vectors and we assume that the torus T^4 is obtained by factoring $\mathbb{R}^4$ over the lattice $\mathbb{Z}^4$ in the standard position. The above tori intersect at three points:

$$\text{at } (0, 0, 0, 0) : t_1, t_2, t_3, t_4 \qquad \text{at } (0, 1/2, 0, 1/2) : t_1, t_3 \qquad \text{at } (1/2, 0, 1/2, 0) : t_2, t_4,$$

yielding numbers of intersections $n_2 = 2$ and $n_4 = 1$. Finally, combining (21) with (13) and using the values of c_i found above, we obtain the Betti numbers for $T^N\backslash A$ given in table 1.

The computation for other two-dimensional patterns does not differ qualitatively from the case of Ammann–Beenker tiling. The only exception is the Penrose tiling, which depends on an extra parameter γ [25]. For a generic value of γ, the arrangement A consists of ten affine tori, but when $\gamma \in \mathbb{Z}[\tau]$ (or, in other words, $\gamma = a + b\tau$), where $\tau = (5^{1/2} - 1)/2$, pairs of parallel thickened tori knit together. This is illustrated in figure 4, in which a part of Penrose tiling with $\gamma = 5\tau - 3$ is shown. Since the tiling in figure 4 is obtained by a singular cut, position of certain vertices is undefined (the affected tiles are shaded). The ambiguously tiled regions are aligned along ten straight lines, corresponding to ten thickened affine tori of Y_r. However, with increasing r, each pair of parallel lines will form a single band on the plane

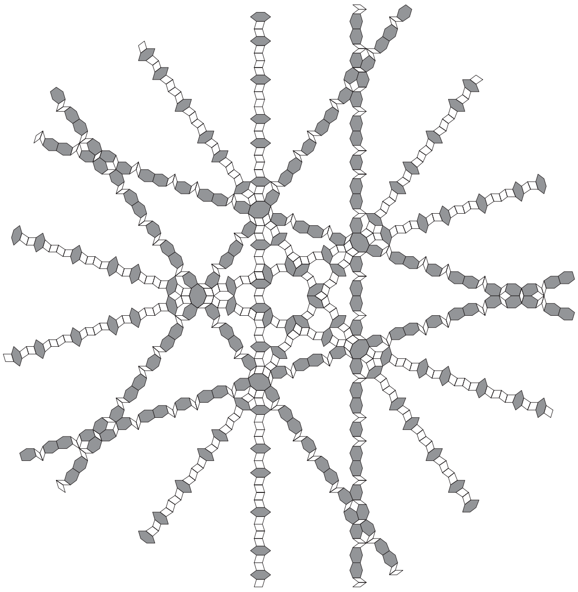

Figure 4. Generalized Penrose tiling ($\gamma = 5\tau - 3$) in a singular position. For illustrative purposes only the ambiguously tiled regions (shaded) and the tiles connecting them to infinite bands are shown.

of the cut. As a result, the arrangement A consists of only five affine tori, all intersecting at the same point. An infinitesimal variation of γ causes displacement of tori making up Y_r in the direction transversal to the cut, and they do not knit together anymore. This peculiarity of the values $\gamma \in \mathbb{Z}[\tau]$ was first observed in [26]. Note, however, that we do not see any anomalous behaviour of the cohomology groups for two other classes of γ, reported in [27], namely $\gamma \in \pm 1/3 + \mathbb{Z}[\tau]$ and $\gamma \in 1/2 + \mathbb{Z}[\tau]$.

7. Icosahedral Ammann–Kramer tiling

The atomic surface of the Ammann–Kramer tiling is the triacontahedron obtained as the projection of the unit cube onto $E_\perp$. Each of 30 faces of the atomic surface gives rise to an R-dense set of points on a plane in the corresponding singular cut. The singular cut crossing a face of the triacontahedron always crosses the opposite face as well. As a result, the thickened affine tori (4) corresponding to the opposite faces knit together. Note also that since a singular cut crossing the triacontahedron at its vertex also crosses it at all faces, all resulting 15 thickened tori have a nonempty common intersection. They can also be thinned down to 15 four-dimensional affine tori, as explained at the end of the appendix. These tori are perpendicular to the two-fold symmetry axes. They intersect each other at 46 two-dimensional tori, which form three orbits under the action of the point symmetry group of the arrangement. Two orbits of 15 elements consist of the tori parallel to the two-fold symmetry axes, one orbit of 10 elements comprises the tori parallel to the three-fold axes, and the remaining orbit includes six tori parallel to five-fold axes. There are 32 intersection points, forming two orbit of 15 points and two exceptional points through which pass all four-dimensional tori. Since the length of maximal chains of the intersection poset equals 3, there are only three non-zero columns in the associated Mayer–Vietoris double complex. The corresponding

Table 2. Multiplicities of irreducible representations of $\boldsymbol{I} \times \mathbb{Z}_2$ for the elements of the spectral sequence E^2 for the Ammann–Kramer tiling.

Irrep	Dimension	$E^2_{0,4}$	$E^2_{0,3}$	$E^2_{1,2}$	$E^2_{0,2}$	$E^2_{1,1}$	$E^2_{2,0}$	$E^2_{0,1}$	$E^2_{1,0}$	$E^2_{0,0}$
A	1	1			1		1			1
A'	1			1			1			
T_1	3		4		2	3		1		
T_1'	3					2	2			
T_2	3		4		2	3		1		
T_2'	3					2	2			
G	4	1	4	1	3	4	1			
G'	4			1		2	3			
H	5	2	4	1	3	6	2			
H'	5			2		2	4			

homology spectral sequence thus necessarily collapses at the E^3-term. But, as we shall see, the only remaining nontrivial differential $\partial_2 : E^2_{2,0} \to E^2_{0,1}$ vanishes because of the symmetry considerations, and the spectral sequence collapses already at the E^2-term

The idea to use the symmetry of the pattern stems from the observation that there is a naturally defined right action of the space symmetry group of $T^N\backslash A$ on the cohomologies of this space. Similarly, one can define a left action of this group on the homology groups of A. This action can be continued onto the simplicial resolution space A^Δ and hence on the entire Mayer–Vietoris double complex. Since the differentials of the associated homology spectral sequence commute with the action of the symmetry group, the group action is also defined on all terms of the spectral sequence. It is natural to decompose the elements of the spectral sequence in the direct sum of irreducible representations of the symmetry group (assuming that the homologies with coefficients in $\mathbb{R}$ are considered). The result of such decomposition is shown in table 2. The symmetry of the arrangement A is that of the body-centred icosahedral six-dimensional lattice (note that the symmetry of A is higher than that of the tiling itself). The space group factored over the translations of the cubic lattice is isomorphic to $\boldsymbol{I} \times \mathbb{Z}_2$. We use the notation of [28] for the irreducible representations of $\boldsymbol{I}$, while the symmetric and antisymmetric representation with respect to $\mathbb{Z}_2$ part are distinguished by adding a prime to the symbol of antisymmetric representation.

As may be seen from table 2, no irreducible representation occurs in both $E^2_{2,0}$ and $E^2_{0,1}$. Hence, no nontrivial differential map can exist between these groups. As there are no other potentially nontrivial differentials at E^2, the spectral sequence collapses at the E^2-term. The elements of E^2 thus correspond to the summands of the graded modules associated with the homology groups $H_*(A)$. Since the inclusion maps of the corresponding filtration of $H_*(A)$ commute with the action of the symmetry group, table 2 also defines the decomposition of $H_*(A)$ into irreducible representations. Recall, however, that our goal is to compute the cohomology groups of $T^6\backslash A$, which are related with $H_*(A)$ by the exact sequence (12). The symmetry group acts on all elements of (12) (the right action on the homology groups should be defined as the left action of the inverse element), and this action commutes with the maps of (12). Hence, projections of the exact sequence (12) onto irreducible representations of the symmetry group can be considered independently. Table 3 shows the decomposition of various terms of (12) into irreducible representations (note that the maps β^k are zero for $k \geqslant 4$).This decomposition together with the data from table 2 gives the final answer for the cohomology groups of $T^6\backslash A$ for the Ammann–Kramer tiling, as shown in table 4. The Betti numbers obtained this way differs by one in dimensions 2 and 3 from those reported in [15].

Table 3. Multiplicities of irreducible representations of $\boldsymbol{I} \times \mathbb{Z}_2$ for the elements of the exact sequence (12) for the Ammann–Kramer tiling.

		Irrep multiplicities									
Module	Dimension	A	A'	T_1	T_1'	T_2	T_2'	G	G'	H	H'
$\mathrm{Im}(\beta^0)$	1	1									
$\mathrm{Im}(\beta^1)$	6			1		1					
$\mathrm{Im}(\beta^2)$	6			1		1					
$\mathrm{Im}(\beta^3)$	2	2									
$H^0(T^6)$	1	1									
$H^1(T^6)$	6			1		1					
$H^2(T^6)$	15			1		1		1		1	
$H^3(T^6)$	20	2						2		2	
$H^4(T^6)$	15			1		1		1		1	
$H^5(T^6)$	6			1		1					
$H^6(T^6)$	1	1									

Table 4. Betti numbers and multiplicities of irreducible representations of $\boldsymbol{I} \times \mathbb{Z}_2$ for cohomology groups of $T^6\backslash A$ for the Ammann–Kramer tiling.

		Irrep multiplicities									
Cohomology group	Betti number	A	A'	T_1	T_1'	T_2	T_2'	G	G'	H	H'
$H^0(T^6\backslash A)$	1	1									
$H^1(T^6\backslash A)$	12	1		1		1				1	
$H^2(T^6\backslash A)$	72		1	5		5		3	1	3	2
$H^3(T^6\backslash A)$	181	4	1	4	4	4	4	7	5	10	6

8. Summary and discussion

In this paper, we have shown that if the 'atomic surface' of a quasiperiodic pattern satisfies the rationality condition, the cohomology ring of its continuous hull is isomorphic to that of a complement of a torus to an arrangement of thickened affine subtori. This fact can be used to compute the cohomology of the hull. The calculations confirm the previously obtained results in most cases, with exception of the generalized Penrose tiling and Ammann–Kramer tiling. The reason for these discrepancies is still unclear (although there are indications [29] that this may be due to a computational error in [15, 27]).

It should be emphasized that the method of this paper could be applied to other homotopy invariants of the hull as long as they correspond to continuous functors from the homotopy category. In particular, the K-theory of the hull should be isomorphic to that of $T^N\backslash A$. This is an important observation since K-groups of the hull are used to label the gaps in the spectra of quasiperiodic potentials [30, 31]. The isomorphism between K-groups of the hull and of $T^N\backslash A$ could provide us with a more intuitive geometric view of the nature of the gaps and spectral projections.

The cohomologies of $T^N\backslash A$ also provide a way for classification of topological matching faults in quasicrystals [32]. This can be illustrated by the following example. Let us consider a large spherical patch of quasicrystal containing no matching faults near the surface. The question arises: is it possible to tell just by looking at the surface that there are matching faults in the interior of the patch? In some instances the answer may be positive. Indeed, as the surface layer is free of matching faults, one can define the map $S^2 \to T^N\backslash A$, where S^2 represents the surface of the patch. If there are no matching faults in the entire patch, this map can be continued to the three-dimensional disc. Clearly, if the homotopy type of the map $S^2 \to T^N\backslash A$ is nontrivial, such continuation is not possible. Hence, the elements of $\pi_2(T^N\backslash A)$ correspond to irremovable point-like matching faults; in the same manner, the linear defects are characterized by the elements of $\pi_1(T^N\backslash A)$. Therefore, each element of cohomology groups of $T^N\backslash A$ defines an integer-valued function on the matching faults through the dual of Hurewicz map $H^n(T^N\backslash A) \to \hom(\pi_n(T^N\backslash A), \mathbb{Z})$. These values could be interpreted as 'topological charges' of matching faults.

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Appendix

This appendix contains the proof of corollary 2. To begin with, let us consider the compact space Y_r as a polyhedron in a local PL topology of T^N. Then, there exists a regular neighbourhood of Y_r in T^N, which we denote by N_{Y_r}. The complement to its interior $T^N\backslash\mathring{N}_{Y_r}$ is a subspace of $T^N\backslash Y_r$, and could also be considered as a subspace of X_r. Owing to the properties of regular neighbourhoods, one can define a deformation retraction of $\rho : T^N\backslash Y_r \to T^N\backslash\mathring{N}_{Y_r}$. The question arises, whether it is possible to extend ρ on X_r or in other words whether there exists a deformation retraction ρ' making the following diagram commutative:

$$\begin{tikzcd} X_r \arrow[dr, dotted, "\rho'"] & \\ T^N\backslash Y_r \arrow[u, "\mu"] \arrow[r, "\rho"] & T^N\backslash\mathring{N}_{Y_r} \end{tikzcd} \tag{A.1}$$

The answer depends on the topology of the embedding of Y_r in T^N, because in general the metric completion modifies the homotopy type of the complement (e.g., for the complements to manifolds of codimension bigger than 1). The following condition is sufficient for extension of ρ' on X_r:

Lemma. *If any point $y \in Y_r$ has a simplicial neighbourhood N_y in T^N such that $N_y \bigcap \{T^N\backslash Y_r\}$ is collapsible in a finite number of steps on $\partial N_y \bigcap (T^N\backslash Y_r)$ then there exists a deformation retraction $\rho : T^N\backslash Y_r \to T^N\backslash\mathring{N}_{Y_r}$ for which the diagram (A.1) can be completed by ρ'.*

Proof. Let (K, L) be the triangulations of (N_{Y_r}, Y_r), which exist by virtue of the simplicial neighbourhood theorem [33]. By the condition of the lemma, for each vertex a of L there exists a collapse

$$N(a, K)\backslash N(a, L) \searrow \partial N(a, K)\backslash\partial N(a, L), \tag{A.2}$$

where $N(a, K)$ and $N(a, L)$ stand for simplicial neighbourhoods of a in K and L, respectively. The composition of collapses (A.2) for all vertices of L gives a collapse

$$N_{Y_r}\backslash Y_r \searrow \partial N_{Y_r}, \tag{A.3}$$

yielding a deformation retraction $\rho : T^N\backslash Y_r \to T^N\backslash\mathring{N}_{Y_r}$. As a composition of finite number of simplicial maps of finite simplicial complexes, the collapse (A.3) satisfies the Lipschitz condition. Hence, any Cauchy sequence in $T^N\backslash Y_r$ remains Cauchy during the deformation retraction ρ, which allows us to extend ρ to the metric completion of $T^N\backslash Y_r$. $\square$

The task is now to show that the set Y_r satisfies the condition of the above lemma for large enough r. According to the remarks made at the end of section 2, it suffices to consider the case when Y_r is a union of thickened tori $t_{r,i}$ (4). Let us introduce a local coordinate system on T^N by treating points in a neighbourhood of $a \in T^N$ as vectors $\mathbf{x} \in \mathbb{R}^n$ with a corresponding to the origin (the space $\mathbb{R}^N$ can be thought of as a universal covering space of T^N). Consider a thickened torus $t_{r,i}$ and let $(\mathbf{n}_i, \mathbf{k}_i)$ be the corresponding unit vectors as defined in section 2. If a is an interior point of $t_{r,i}$ then the equation of $t_{r,i}$ in the neighbourhood of a is

$$\mathbf{x} \cdot \mathbf{k}_i = 0. \tag{A.4}$$

If a lies at the boundary of $t_{r,i}$ then one has to add one of the following inequalities to condition (A.4):

$$\mathbf{x} \cdot \mathbf{n}_i \geqslant 0 \qquad \text{or} \qquad \mathbf{x} \cdot \mathbf{n}_i \leqslant 0. \tag{A.5}$$

Let now a be an arbitrary point of Y_r. It belongs to $t_{r,i}$ for $i \in I' \subseteq I$ and lies at the boundary of $t_{r,i}$ for $i \in I'' \subseteq I'$ (the set I'' may be empty). One can choose a neighbourhood of a in the form $B_\epsilon = B_\epsilon^\parallel \times B_\epsilon^\perp$, where $B_\epsilon^\parallel$ and $B_\epsilon^\perp$ are PL ϵ-balls in $E_\parallel$ and $E_\perp$ correspondingly. Our goal is to give an explicit construction of the collapse $B_\epsilon \bigcap (T^N\backslash Y_r) \searrow \partial B_\epsilon \bigcap (T^N\backslash Y_r)$. We begin by cutting B_ϵ by hyperplanes $\{\mathbf{x}\cdot\mathbf{k}_i = 0 | i \in I'\}$ and $\{\mathbf{x}\cdot\mathbf{n}_i = 0 | i \in I''\}$. The resulting cells together with all their faces form a cell complex G with the underlying space $|G| = B_\epsilon$. It is pertinent to note that $B_\epsilon \bigcap Y_r$ corresponds to a subcomplex H of G. Furthermore, the complex G is in fact a product of two cell complexes $G = G^\parallel \times G^\perp$ obtained by cutting of $B_\epsilon^\parallel$ and $B_\epsilon^\perp$ by the hyperplanes orthogonal to $\mathbf{n}_i$ and $\mathbf{k}_i$, respectively. For any cell $C \in G^\parallel$ except of maybe one, which we denote by C_0, the space $B_C = C \times B_\epsilon^\perp$ is cut by one or more of the hyperplanes (A.4). Hence, the complement to its intersection with Y_r is collapsible to the analogous complement of its boundary: $B_C \big\backslash \big(B_C \bigcap Y_r\big) \searrow \partial B_C \big\backslash \big(\partial B_C \bigcap Y_r\big)$. Performing the collapses in the order of decreasing dimension of cells yields either $\partial B_\epsilon \big\backslash \big(\partial B_\epsilon \bigcap Y_r\big)$ if the exceptional cell C_0 does not exist or $\big(\partial B_\epsilon \bigcup B_{C_0}\big) \big\backslash \big(\big(\partial B_\epsilon \bigcup B_{C_0}\big) \bigcap Y_r\big)$ otherwise. Because the interiors of both $B_{C_0} \big\backslash \big(B_{C_0} \bigcap Y_r\big)$ and $\partial B_\epsilon \bigcap \big(B_{C_0} \big\backslash \big(B_{C_0} \bigcap Y_r\big)\big)$ are open discs, one more collapse reduces the latter case to the former, which proves that the union of thickened tori (4) satisfies the condition of the lemma.

It remains to construct an arrangement of thickened affine tori A in T^N such that $A \subset Y_r$ and that the natural inclusion $\nu : T^N\backslash Y_r \to T^N\backslash A$ is a homotopy equivalence. Actually it suffices to show that $Y_r \searrow A$, because then the regular neighbourhood of Y^r in T^N is also a regular neighbourhood of A (see corollary 3.29 from [33]). To begin with, consider an intersection of a singular cut with Y_r, which is a finite union of thickened hyperplanes. As r increases, some faces of the resulting polyhedron may disappear, but for r big enough the shape of the polyhedron eventually stabilizes (see figure 5). Further still, the value of r for which the stabilization occurs is uniformly bounded by some finite positive r_0. This follows from the observation that the intersection of a singular cut with Y_r is defined up to translation by the set of faces of ∂S through which the cut passes and that ∂S has a finite number of faces. Consider now the local structure of Y_r for $r \geqslant r_0$. Any point at the boundary of Y_r has a neighbourhood B_ϵ in which Y_r is locally defined by conditions (A.4) and (A.5). The stability of the shape of the intersection of Y_r with a singular cut implies that small variations of r correspond to a local parallel translations of the boundary of Y_r. Owing to the compactness of the boundary of Y_r one can choose a finite covering of it by neighbourhoods B_ϵ such that the boundaries of $Y_{r+\delta}$ and $Y_{r-\delta}$ are contained within it for some $\delta > 0$. An appropriate triangulation of these

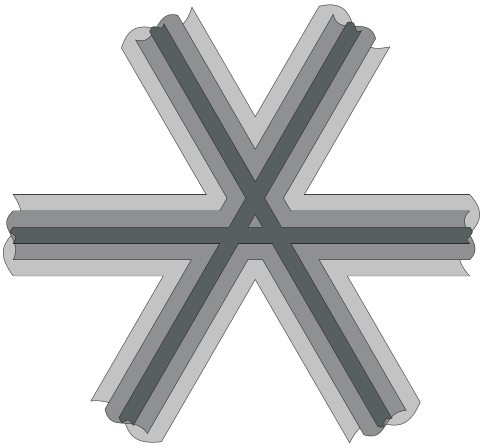

Figure 5. Intersection of a singular cut with Y_r for different values of r. The shape of the resulting union of thickened hyperplanes stabilizes with increasing r.

neighbourhoods thus defines a collapse $Y_{r+\delta} \searrow Y_{r-\delta}$. Hence, for any $r > r_0$ one has $Y_r \searrow Y_{r_0}$ and the arrangement of thickened tori $A = Y_{r_0}$ satisfies conditions of corollary 2.

The last statement of corollary 2 follows from the commutativity of the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & X_{r_{n+1}} \arrow[r, "\iota_n"] & X_{r_n} \arrow[r, "\iota_{n-1}"] & X_{r_{n-1}} \arrow[r] & \cdots \\
\cdots \arrow[r] & T^N\backslash Y_{r_{n+1}} \arrow[u, "\mu_{n+1}"] \arrow[r, "\iota'_n"] \arrow[dr, "\nu_{n+1}"] & T^N\backslash Y_{r_n} \arrow[u, "\mu_n"] \arrow[r, "\iota'_{n-1}"] \arrow[d, "\nu_n"] & T^N\backslash Y_{r_{n-1}} \arrow[u, "\mu_{n-1}"] \arrow[r] \arrow[dl, "\nu_{n-1}"] & \cdots \\
 & & T^N\backslash A & &
\end{tikzcd} \tag{A.6}$$

It should be pointed out here that in some cases the thickened tori constituent the arrangement Y_{r_0} can be 'thinned down'. In more exact terms, Y_{r_0} can be collapsed to an arrangement of ordinary affine tori, which may be substituted for A in corollary 2. In particular, 'thinning down' is possible when all thickened tori in Y_{r_0} have a nonempty intersection. in this case, one can choose a point in the common intersection of thickened tori and make all affine tori of A pass through this point, which can also be set as the origin. Clearly, each nonempty intersection in A is also so over the rational numbers. Since no two rational points in T^N can belong to the same cut, the hyperplanes in $E_\parallel$ obtained by intersection with A always have a nonempty intersection, and Y_{r_0} can be collapsed to A. The quasiperiodic patterns obeying substitution rules also fall in this category; in this case, A may be thought of a result of 'infinite deflation' applied to Y_{r_0}. There are other cases when Y_{r_0} can be 'thinned down' including, among others, the generalized Penrose tiling. It remains unclear, however, whether this possibility is the common property of all patterns with polyhedral atomic surfaces satisfying the rationality condition.

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